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On the solvability of the Birkhoff interpolation problem ☆

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Abstract

A univariate Birkhoff interpolation problem is considered and the smallest N, namely $\bar{N}$, for which the Birkhoff interpolation problem is always solvable is given. In particular, for Pólya matrices E with n ones we obtain $\bar{N} \leqslant n + p$, where p is the number of odd supported sequences in E, and show that the preceding bound is the best possible.

Keywords: Birkhoff interpolation; Rectangular Vandermonde matrices

1. Introduction

An $m \times d$ matrix $E = [e_{ij}]_{i=1,\, j=0}^{m \;\; d-1}$ is an *incidence matrix* if its entries e_{ij} are 0 or 1. By $\chi(E)$ we denote the set of real m-tuples (called *knots*) $\chi(E) = \{(x_1, x_2, \ldots, x_m) | x_1 < x_2 < \cdots < x_m\}$.

A univariate *algebraic Birkhoff interpolation problem*, $(X, E, C, \mathbb{P}_{N-1})$, consists of a vector $X \in \chi(E)$, an incidence matrix E with exactly n ones, a real data vector C built up by n values $(c_{ij} | e_{ij} = 1)$, and the space $\mathbb{P}_{N-1}$ of real polynomials of degree at most $N - 1$.

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The purpose of an algebraic interpolation problem is to find a polynomial $P(x)$ in $\mathbb{P}_{N-1}$ that satisfies the conditions

$$P^{(j)}(x_i) = c_{ij}, \quad e_{ij} = 1. \tag{1}$$

Writing polynomial $P(x) \in \mathbb{P}_{N-1}$ in the form $P(x) = \sum_{k=0}^{N-1} a_k \frac{x^k}{k!}$, conditions (1) determine a linear system with n equations and N unknowns $A_N(E,X) \cdot A = C$, where $A = [a_0, a_1, \ldots, a_{N-1}]^T$ is the vector of unknowns and $A_N(E,X)$ is the $n \times N$ coefficients matrix:

$$A_N(E,X) = \left[\frac{x_i^{-j}}{(-j)!} \quad \frac{x_i^{1-j}}{(1-j)!} \quad \frac{x_i^{2-j}}{(2-j)!} \quad \cdots \quad \frac{x_i^{N-1-j}}{(N-1-j)!} \right]_{e_{ij}=1} \tag{2}$$

with the convention $\frac{x_i^h}{h!} = 1$ if $h = 0$ and $\frac{x_i^h}{h!} = 0$ if $h < 0$. Consequently, we have

$$\text{rank}\, A_N(E,X) + \rho = N, \tag{3}$$

where ρ is the dimension of the subspace of polynomials $P(x) \in \mathbb{P}_{N-1}$ annihilated by (E,X).

In what follows, the case when $N = n$ will be called *square-case*. The pair (E,X) is said to be *regular* if for each choice of the data c_{ij} there exists a unique solution $P(x) \in \mathbb{P}_{N-1}$ of (1) being $N = n$. Schoenberg in [5] posed the problem of determining all those matrices E for which the pair (E,X) is always regular. Such matrices E are said to be *order regular*, and the remaining matrices *order singular*. Note that if $N \geqslant 1$, then the condition

$$\text{rank}\, A_N(E,X) = n, \quad \text{for all } X \in \chi(E) \tag{4}$$

holds if and only if for each $X \in \chi(E)$ and for any choice of the data c_{ij} there exists at least one solution $P(x) \in \mathbb{P}_{N-1}$ of (1). In this case, from (3) it follows that the dimension of the subspace of polynomials $P(x) \in \mathbb{P}_{N-1}$ annihilated by (E,X) is exactly $N - n$.

By *length* of E we mean the largest $\ell \geqslant 1$ for which there exists an i with $e_{i,\ell-1} = 1$. If $N \geqslant 1$ and rank $A_N(E,X) = n$ for some choice of $X \in \chi(E)$ then $N \geqslant \max\{\ell, n\}$. The numbers $M_j = \sum_{j'=0}^{j} \sum_{i=1}^{m} e_{ij'}$, $j \geqslant -1$, are called *Pólya constants* of E. Note that conditions (i) $M_j \geqslant j+1$, if $j = 0, 1, \ldots, n-1$, and (ii) $M_j \geqslant j+1$, if $j = 0, 1, \ldots, \ell - 1$, are equivalent, and are called *Pólya condition*. If E satisfies the Pólya condition then E is *normal*. That is, $\ell \leqslant n$.

A sequence of ones of the ith row of E is *supported* if when (i,j) is the position of the first 1 of the sequence, this implies that there exist two ones: $e_{i_1,j_1} = e_{i_2,j_2} = 1$ with $i_1 < i < i_2, j_1 < j$, and $j_2 < j$. Then, we have [1].

Theorem 1.1. *If E satisfies the Pólya condition and contains no odd supported sequences, then E is order regular.*

A review on algebraic Birkhoff interpolation can be found in Lorentz et al. [4].

Let p be the number of odd supported sequences of matrix E. If $j \geqslant -1$, the number of odd supported sequences of E having its first element $e_{i',j'} = 1$ with $j' \leqslant j + 1$ will be denoted by S_j. We have $0 = S_{-1} \leqslant S_0 \leqslant S_1 \leqslant \cdots \leqslant S_{\ell-2} = S_j = p$ for each $j \geqslant \ell - 2$. The number

$$\tau = \max_{-1 \leqslant j \leqslant \ell-1} \{ j + 1 - M_j - S_j \} \tag{5}$$

will be called *almost-Pólya defect* of E. The value $j = -1$ is included in (5) only to assure $\tau \geqslant 0$. Notice that each Pólya matrix has $\tau = 0$.

Our interest in this note is focused on the study of the smallest N for which (4) holds. Namely, $\bar{N}$, and called *solvability number* of E. Notice that $\bar{N} \geqslant n$, and $\bar{N} = n$ if and only if E is order regular. Moreover, the set $\{N | N \geqslant \bar{N}\}$ coincides with the set of N's for which condition (4) holds. In this paper, we will prove that $\bar{N} \leqslant n + p + \tau$ and we will also establish that the preceding bound is the best possible.

2. The main result

Taking into account the preceding considerations, we can now state and prove our main result.

Theorem 2.1. (a) *Let E be an incidence matrix with n ones, p odd supported sequences and almost-Pólya defect τ. If $\bar{N}$ is the solvability number of E, then*

$$\bar{N} \leqslant n + p + \tau. \tag{6}$$

(b) *Given three integer numbers $n \geqslant 1, p \geqslant 0, \tau \geqslant 0$, with $n \geqslant p + 2$ if $p \neq 0$, there exists an incidence matrix E with n ones, p odd supported sequences and almost-Pólya defect τ, for which inequality* (6) *becomes equality. Furthermore, if $\tau = 0$ and $n \geqslant 2p + 1$ we can select E being a Pólya matrix.*

Part (b) of this theorem shows that inequality (6) cannot, in general, be improved. Note that condition $n \geqslant p + 2$ if $p \neq 0$ is necessary.

Proof. Before giving the proof of the theorem, we state and prove the following lemma.

Lemma 2.1. *Let $E = [e_{ij}]_{i=1,j=0}^{m\ d-1}$ be an incidence matrix with no odd supported sequences, and let $L = [\ell_{ij}]_{i=1,j=0}^{r\ d-1}$ be a Lagrange matrix (with ones only in its first column). Then, there exists an $i_0 \in \{0, 1, 2, \ldots, m\}$ such that matrix $E^L_{i_0,i_0+1}$ which is obtained from E by inserting matrix L between the i_0th row and the $(i_0 + 1)$th row of E, has no odd supported sequences.*

Proof. We consider an element $e_{i^*,j^*} = 1$ in the smallest possible column of E (having $j = j^*$ minimum). The i^*th row of E is $v = [e_{i^*,0}, e_{i^*,1}, \ldots]$, and its first 1 is $e_{i^*,j^*} = 1$. If

E_1 and E_2 are the matrices obtained from the first i^*-1 rows and last $m-i^*$ rows of E, respectively, then

$$E = \begin{bmatrix} E_1 \\ v \\ E_2 \end{bmatrix}, \quad E^L_{i^*-1,i^*} = \begin{bmatrix} E_1 \\ L \\ v \\ E_2 \end{bmatrix}, \quad E^L_{i^*,i^*+1} = \begin{bmatrix} E_1 \\ v \\ L \\ E_2 \end{bmatrix}.$$

In what follows, we will prove that $E^L_{i^*-1,i^*}$ or $E^L_{i^*,i^*+1}$ cannot have odd supported sequences. Clearly, neither of them can have an odd supported sequence arising from E_1, E_2 or L (for E_1 and E_2 this fact is due to the fact that E has no odd supported sequences and $e_{i^*,j^*}=1$ is the first one appearing in E). Hence, $E^L_{i^*-1,i^*}$ and $E^L_{i^*,i^*+1}$ can only have odd supported sequences arising from v. Therefore, if $i^*=1$ or $i^*=m$ we are done. Finally, assume $1<i^*<m$. Let $e_{i_1,j_1}=1$ and $e_{i_2,j_2}=1$ be two entries in the smallest possible column of E_1 and E_2, respectively. We have $1\leqslant i_1<i^*<i_2\leqslant m$, $j_1\geqslant j^*$ and $j_2\geqslant j^*$. Since all supported sequences of $E^L_{i^*-1,i^*}$ and $E^L_{i^*,i^*+1}$ must arise from row v, it follows that if $j_1\leqslant j_2$, then $E^L_{i^*-1,i^*}$ cannot have odd supported sequences, and the same holds for $E^L_{i^*,i^*+1}$ when $j_1\geqslant j_2$, and we are done. □

Firstly, we prove part (a) of Theorem 2.1. Let E be the incidence matrix of part (a), having m rows and length ℓ. Let $I_r: e_{i_r,j_r}=e_{i_r,j_r+1}=\cdots=e_{i_r,j'_r}=1; r=1,\dots,p$, be the distinct odd supported sequences of E. We define a new matrix E' as the incidence matrix obtained from E by replacing entries $e_{i_r,j_r-1}=0$ $(1\leqslant r\leqslant p)$ by $e_{i_r,j_r-1}=1$. E' has m rows, length ℓ, $n+p$ ones, and for $j\geqslant 0$, $M'_j=M_j+S_j$, where M'_j are the Pólya constants of E'. Moreover, it is easy to see that E' has no odd supported sequences (note that the odd supported sequences have been transformed into even sequences). Thus, we can apply Lemma 2.1 to $E=E'$ and L, the Lagrange matrix with τ rows. We obtain a new matrix $E''=(E')^L_{i_0,i_0+1}$ with no odd supported sequences, $n+p+\tau$ ones and length ℓ. Furthermore, from the definition of τ, if $j=0,1,\dots,\ell-1$ then $M''_j=\tau+M'_j=\tau+M_j+S_j\geqslant j+1$, where M''_j are the Pólya constants of E''. Hence, E'' is a Pólya matrix with no odd supported sequences. From Theorem 1.1 we know that E'' is order regular; or equivalently, $\operatorname{rank} A_{n+p+\tau}(E'',X'')=n+p+\tau$ for all $X''\in\chi(E'')$. It follows that $\operatorname{rank} A_{n+p+\tau}(E,X)=n$ if $X\in\chi(E)$, since all rows of $A_{n+p+\tau}(E,X)$ are also rows of $A_{n+p+\tau}(E'',X'')$ for an appropriate choice of X'' (X'' is built up by adding τ components to X). Hence, we have obtained (4) for number $N=n+p+\tau$, and part (a) is proved.

Next, we prove part (b). Case $p=0$ is easy, since for the one-row matrix $E=[e_{1j}]_{j=0}^{d-1}$ defined by $e_{1j}=1$ if $\tau\leqslant j\leqslant\tau+n-1$, and $e_{1j}=0$ otherwise, we get $\operatorname{rank} A_{n+\tau-1}(E,X)<n$ for all $X\in\chi(E)$. Note that if $\tau=0$ then E is a Pólya matrix.

Now let $n\geqslant 1$, $p\geqslant 1$ and $\tau\geqslant 0$ be given with $n\geqslant p+2$. Let k be an auxiliary integer with

$$1 \leqslant k \leqslant 2p, \quad \text{and} \quad q = n - p - 2k \geqslant 0. \tag{7}$$

For example, $k = 1$ satisfies these assumptions. We consider matrix E defined by blocks $E = [E'|E'']$, where (i) $E' = [e'_{ij}]_{i=1,j=0}^{3\ \ q+\tau-1}$ is a $3 \times (q+\tau)$ matrix with $e'_{2j} = 1$ if $0 \leqslant j \leqslant q-1$ and $e'_{ij} = 0$ otherwise, and (ii) E'' is the incidence matrix

$$E'' = \left[\begin{array}{cccccccc|c} 1 & 1 & 1 & 1 & \ldots\ldots. & .. & .. & & \\ 0 & 1 & 0 & 1 & \ldots\ldots. & 0 & 1 & & 0 \\ 1 & 1 & 1 & 1 & \ldots\ldots & .. & .. & & \end{array}\right], \tag{8}$$

where the 1st and 3rd rows have ones on the first k columns and zeros on the other columns, and the 2nd row has p groups $[0, 1]$. Matrices like (8) are useful for the study of the lowest possible rank of $A_N(E, X)$ in the square-case ([2] or [4, p. 15]). E has $q + p + 2k = n$ ones and p odd supported sequences. It is easy to see that the almost-Pólya defect of E is τ, and E is a Pólya matrix if and only if $\tau = 0$ and $k \geqslant \frac{p}{2}$.

We will see that rank $A_N(E, X) < n$ for $X = (-1, 0, 1)$ and $N = n + p + \tau - 1$. For each $t = 0, 1, \ldots, p-1$, let Q_t be a polynomial satisfying $Q_t^{(q+\tau)} \equiv P_t$, where $P_t(x) = (x^2-1)^k x^{2t}$, and also satisfying $Q_t(0) = Q'_t(0) = \cdots = Q_t^{(q-1)}(0) = 0$. Since P_t is annihilated by (E'', X), Q_t is annihilated by (E, X). On the other hand, all polynomials $x^q, x^{q+1}, \ldots, x^{q+\tau-1}$ are also annihilated by (E, X). Hence, we have $p + \tau$ polynomials with distinct degree $\leqslant N - 1$, and annihilated by (E, X). From (3) it follows that rank $A_N(E, X) < n$.

We have just proved the first statement of part (b) of Theorem 2.1, by setting, for example, $k = 1$. For the second statement of part (b) we take $k = [\frac{p+1}{2}]$. Since $n \geqslant 2p + 1$, this k also satisfies the assumptions (7), and from $\tau = 0$ and $k \geqslant \frac{p}{2}$, E is a Pólya matrix. □

3. Concluding remarks

In this note, we have obtained the best possible upper bound for the solvability number of an incidence matrix E, in terms only of its number of ones, its number of odd supported sequences and its almost-Pólya defect. For Pólya matrices E, Theorem 2.1 states that $\bar{N} \leqslant n + p$, this bound being the best possible.

Finally, we give some examples of application of the main result. For matrices

$$E_1 = \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 1 & 1 & \mathbf{0} \\ 0 & 1 & 0 & 0 & 0 & \end{array}\right], \quad E_2 = \left[\begin{array}{cccccc|c} 0 & 1 & 1 & 0 & 0 & 1 & \\ 1 & 1 & 1 & 0 & 0 & 1 & \mathbf{0} \\ 1 & 0 & 0 & 1 & 1 & 1 & \\ 0 & 1 & 1 & 0 & 0 & 0 & \end{array}\right],$$

$$E_3 = \left[\begin{array}{cc|c} 1 & 0 & \\ 0 & 1 & \\ 0 & 1 & \mathbf{0} \\ 1 & 0 & \end{array}\right]$$

inequality (6) yields $\bar{N}_1 \leqslant 4 + 0 + 1 = 5$, $\bar{N}_2 \leqslant 13 + 2 + 0 = 15$, and $\bar{N}_3 \leqslant 4 + 2 + 0 = 6$, respectively. For matrix E_1, we have $\bar{N}_1 = 5$, since E_1 is not normal. For E_2, note that it has at least one row with exactly one odd supported sequence and it satisfies the strong Pólya condition $M_j \geqslant j + 2$ for $j = 0, 1, \ldots, n-2$. From Lorentz's Theorem in [3] or [4, p. 65] we know that E_2 is order singular, and hence $14 \leqslant \bar{N}_2 \leqslant 15$. Finally, for E_3, by a direct calculation of the rank of $A_5(E_3, X)$, where $X = (-1, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 1)$, we obtain $\operatorname{rank} A_5(E_3, X) = 3 < n$. It follows that for this matrix $\bar{N}_3 = 6$, or in other words, equality in (6) holds again.

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